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بنية جبر هيكي للزمرة الخطية العامة

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بنية جبر هيكي للزمرة الخطية العامة $GL(2,p^n)$

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ملخص البحث

يهـــدف هذا البحث إلى دراسة بنية جبر هيكي $E(N,\lambda)$ للزمره الخطية العامـــة مـــن المرتبة الثانية المرافقة للزوج (N,λ) حيث N هي زمرة المصفوفات التبديلـــية و λ التمثيل الخطي للزمرة N المعرف بدلالة تمثيل الإشارة. في هذا البحـــث يـــتم تحديـــد الأســاس و معاملات البنية لهذا الجبر . كما نستخدم جبر الحدوديـــات لتحليل جبر هيكي عن طريق تحليل العنصر المحايد الى مجموع لعناصر جـــامدة ومـــتعامدة. أخيرا تستخدم تلك النتائج لتمييز العناصر الجامدة ولدراسة عناصر الوحدة في جبر هيكي.

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(1) If $\dim_k E(N,\lambda) \succ 2$ then evaluating $\det(A(x))$ with respect to the bottom row we see that $\det(A(x))=0$, hence x is not unit.

(2) If $\dim_k E(N,\lambda)=2$ then, by 2.6, the characteristic of k must be 2. On the other hand $c_e=0$ implies that $\det(A(x))=2c_0^2(q-1)=0$ and so x is not unit. □

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$e_1 = {}^{-1}\!/_{q+1}(a_0 - 2)$ and $e_2 = {}^{1}\!/_{q+1}(a_0 + (q-1))$ are two orthogonal idempotents in $E(N,\lambda)$ such that $1_{E(N,\lambda)} = e_1 + e_2$. $\square$

Generally we may use the relations in 4.10 together with the action given in 4.11 to characterize the set of idempotents in the algebra $E(N,\lambda)$ as explained in the following proposition

PROPOSITION 6.3 Suppose that $x = \sum_{\alpha\in X} c_\alpha a_\alpha \in E(N,\lambda)$. Then x is idempotent if and only if the coefficients c_α satisfy the following identities

$$c_e(c_e - 1) + \sum_{e\neq\alpha\in X} 2(q-1)c_\alpha^2 = 0$$

$$c_0(2c_e - 1) - \sum_{e\neq\beta\in X}(q-3)c_\beta^2 = 0$$

$$c_\alpha(2c_e - 1) = 0 \ , \forall \alpha \in X \setminus \{0,e\}$$

PROOF Compare the coefficients in $x^2 = x$ and use the fact that $a_\alpha ; \alpha \in X$ are linearly independent in $E(N,\lambda)$. $\square$

Note that the identities in 6.3 hold in particular for the idempotents e_1, e_2 of $E(N,\lambda)$ which are defined in 6.1.

Next we consider the units ; that is the set of invertible elements ,of the Hecke algebra $E(N,\lambda)$. The following gives a partial characterization for the units in $E(N,\lambda)$.

PROPOSITION 6.4 If $x = \sum_{\alpha\in X} c_\alpha a_\alpha \in E(N,\lambda)$ is a unit , then $c_e \neq 0$.

PROOF Relative to the basis $\{a_\alpha ; \alpha \in X\}$, the element $x = \sum_{\alpha\in X} c_\alpha a_\alpha \in E(N,\lambda)$ acts on $E(N,\lambda)$ according to the matrix

$$A(x) = \begin{array}{c} a_e \\ a_0 \\ a_\alpha \\ . \\ a_\beta \end{array} \begin{pmatrix} c_e & 2c_0(q-1) & 2c_\alpha(q-1) & . & 2c_\beta(q-1) \\ c_0 & c_e - c_0(q-3) & -c_\alpha(q-3) & . & -c_\beta(q-3) \\ c_\alpha & 0 & c_e & . & 0 \\ . & . & 0 & . & 0 \\ c_\beta & 0 & 0 & . & c_e \end{pmatrix}$$

Hence x is a unit in $E(N,\lambda)$ if and only if A(x) is nonsingular ,that is if and only if $\det(A(x)) \neq 0$. Now suppose that $c_e = 0$ then we consider two cases :

such that $A_1(X)\Phi_1(X)+A_2(X)\Phi_2(X)=1$. Let $e_i=A_i(a)\Phi_i(a); i=1,2$. Then both E_1 and E_2 are non-zero and $\mathbf{1}_A=e_1+e_2$ is an orthogonal idempotent decomposition in A.

PROOF It is clear from the hypothesis that

$$e_1+e_2=A_1(a)\Phi_1(a)+A_2(a)\Phi_2(a)=1_A$$

Also, since $\Phi_1(a)\Phi_2(a)=\Phi(a)=0$, it follows that

$$e_1e_2=e_2e_1=A_1(a)A_2(a)\Phi_1(a)\Phi_2(a)=0$$

Therefore, $e_1=e_1\mathbf{1}_A=e_1(e_1+e_2)=e_1^2+e_1e_2=e_1^2$. Similarly $e_2^2=e_2$

Now to prove that $e_1\neq 0\neq e_2$, suppose that $e_1=0$, then $A_1(a)\Phi_1(a)=0$ and so $\Phi(X)\mid A_1(X)\Phi_1(X)$. But this implies that $\Phi_2(X)$ divides both $A_1(X)\Phi_1(X)$ and $A_2(X)\Phi_2(X)$, hence

$$\Phi_2(X)\mid A_1(X)\Phi_1(X)+A_2(X)\Phi_2(X)=1$$

which is a contradiction and so $e_1\neq 0$. Similarly $e_2\neq 0$. □

§6. IDEMPOTENTS IN $E(N,\lambda)$.

We shall apply the method described in the previous section to find an orthogonal idempotents decomposition of the identity of Hecke algebra $E(N,\lambda)$. It is well known from the Brauer-Fitting theorem (see [8], 1.4) that such decomposition of a_e gives rise to a decomposition of $E(N,\lambda)$ into a direct sum of algebras and hence to a decomposition for the kG-module $Y(N,\lambda)$.

PROPOSITION 6.1 Suppose that $p \nmid q+1$, and let $e_1=-\frac{1}{q+1}(a_0-2)$, $e_2=\frac{1}{q+1}(a_0+(q-1))\in E(N,\lambda)$. Then e_1,e_2 are orthogonal idemoptents in $E(N,\lambda)$ such that $a_e(=1_{E(N,\lambda)})=e_1+e_2$.

PROOF Take $\alpha=0$ in 4.10. Then we have $a_0^2=2(q-1)-(q-3)a_0$, hence $a_0^2+(q-3)a_0-2(q-1)=0$ is the minimum equation of a_0, and so

6.2 $$X^2+(q-3)X-2(q-1)=(X-2)(X+(q-1))$$

is the minimum polynomial of a_0 with $A_1(X)=(X-2)$ and $A_2(X)=(X+(q-1))$ have no common divisor in k[X]. Now we may apply proposition 5.2 to the identity 6.2 and take $\Phi_1=-\frac{1}{q+1}$, $\Phi_2=\frac{1}{q+1}$ to get

PROPOSITION 4.10 The Hecke algebra $E(N,\lambda)$ is generated as k-algebra by $\{ a_\alpha \ ; \alpha \in X \setminus \{e,0\}\}$ subject to the relations :

$$a_\alpha a_\beta = 0 \qquad \forall \alpha \neq \beta \ ,$$

$$a_\alpha^2 = 2(q-1)a_e - (q-3)a_0 \qquad \forall \alpha \in X \setminus \{e\}$$

□

From the above presentation of the Hecke algebra $E(N,\lambda)$ we deduce the following action of the elements of $E(N,\lambda)$ on the basis elements $a_\beta \ ; \beta \in X$

4.11 $$(\sum_{\alpha \in X} c_\alpha a_\alpha) a_\beta = 2c_\beta(q-1)a_e - c_\beta(q-3)a_0 + c_e a_\beta \qquad \forall \ e \neq \beta \in X$$

§5. POLYNOMIAL ALGEBRA AND IDEMPOTENTS

Constructing idempotents is an essential step towards analyzing any algebra. In this section we shall explain a technique for constructing idempotents in algebras by means of identities in the polynomial algebras. This method shall be applied in the next section to construct idempotents in the Hecke algebra $E(N,\lambda)$ considered in the previous sections. Suppose that A is a finite dimensional k-algebra with an identity 1_A and that a is an element of A . Since A is finite dimensional ,there must be a positive integer n such that the set $\{ 1, a, a^2, a^3, ..., a^n \}$ is linearly dependent ; if n is the least with this property then we have

5.1 $$\sum_{i=0}^{n} c_i a^i = 0 \quad \text{for some (not all zero)} \ c_i \in k$$

Equation 5.1 is called ***the minimum equation for a*** and the polynomial $\Phi(X) = \sum_{i=0}^{n} c_i X^i \in k[X]$ is called ***the minimum polynomial for a***. If $\Omega(X)$ is any other polynomial in k[X] such that $\Omega(a) = 0$ then , using the minimality of $\Phi(X)$, it is easy to see that $\Phi(X) \mid \Omega(X)$. The following shows how to construct , out of $\Phi(X)$, an orthogonal idempotent decomposition of 1_A .

PROPOSITION 5.2 Let A be a finite dimensional k-algebra and let $a \in A$. Suppose that $\Phi(X) = \Phi_1(X)\Phi_2(X)$ is the minimum polynomial of a , where $\Phi_1(X), \Phi_2(X)$ are non-constant polynomial in $k[X]$ with no common divisor in $k[X]$. By Euclid's algorithm there exist $A_1, A_2 \in k[X]$

(In the later case $\alpha \neq \beta$, for if $\alpha = \beta$ then $b + \alpha^2 = 0$, but $b = -\beta = -\alpha$ and so $\alpha^2 = \alpha$, hence $\alpha = 1$.)

$\Leftrightarrow n_a^+ g_\beta n_b^+ g_\alpha = g_0 \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \vee g_0 \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$, hence giving coefficient $\lambda(n_a^+)\lambda(n_b^+)\lambda(\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}) = 1$ in the first case, and the coefficient $\lambda(n_a^+)\lambda(n_b^+)\lambda(\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}) = -1$. When we sum over all $a, b \in F_q^*$ we get $t_{\alpha,\beta,0} = 0$.

Now suppose that $0 \neq \gamma \in X$. Then

$$n_a^+ g_\beta n_b^+ g_\alpha \in g_\gamma N \Leftrightarrow [\, ab + b = b + \beta \ \wedge\ b + \alpha\beta = \gamma(ab + \alpha a)]$$
$$\vee [b + \alpha\beta = ab + \alpha a \ \wedge\ b + \beta = \gamma(ab + a)]$$
$$\Leftrightarrow [\, a = {}^{b+\beta}\!/_{b+1} \wedge b = {}^{\gamma\beta - \gamma a + \gamma\alpha a - \alpha\beta}\!/_{1-\gamma}\,]$$
$$\vee [b = {}^{\alpha a - \alpha\beta}\!/_{1-a} \wedge a = {}^{b+\beta}\!/_{\gamma(b+1)}]$$

Note that $b \neq -1$, $a \neq 1$; otherwise $\beta = 1$. The first case gives the coefficient $\lambda(n^+_{{}^{b+\beta}/_{b+1}})\lambda(n^+_{{}^{\gamma\beta-\gamma a+\gamma\alpha a-\alpha\beta}/_{1-\gamma}})\lambda(\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}) = 1$, for all $b \neq -1$, $a \neq 1$. The second case contributes the coefficient $\lambda(n^+_{{}^{b+\beta}/_{\gamma(b+1)}})\lambda(n^+_{{}^{\alpha(a-\beta)}/_{1-a}})\lambda(\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}) = -1$, for all $b \neq -1$, $a \neq 1$. Clearly these scalars cancel each other when we sum over all such a and b. This proves that $t_{\alpha,\beta,\gamma} = 0$.

Summarizing we have the following

THEOREM 4.9 The structure constants $t_{\alpha,\beta,\gamma}$ of the Hecke algebra $E(N,\lambda)$ are given as follows:

$$t_{\alpha,\beta,\gamma} = \begin{cases} -(q-3) & \text{if } \alpha = \beta = \gamma = 0 \\ 2(q-1) & \text{if } \alpha = \beta \text{ and } \gamma = e \\ 0 & \text{otherwise} \end{cases}$$ □

Now we translate theorem 4.9 into the following presentation for the Hecke algebra $E(N,\lambda)$.

$(a=\gamma^{-1}\beta \wedge b = {}^{\beta-\beta\gamma^{-1}}\!/_{\beta\gamma^{-1}-1}) \vee (a=\beta \ \wedge b = {}^{\beta\gamma-\beta}\!/_{1-\beta\gamma}))$ (note that $\gamma\beta \neq 1$ by our choice of the index set $X \subseteq F_q$.). In the first case we have $n_a^+ g_\beta n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix}\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ which gives the coefficient +1, and in the second case we get $n_a^+ g_\beta n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix}\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ which gives the coefficient -1 , hence we get 0 when we sum over all a,b. This proves the following

PROPOSITION 4.7 If $\beta \neq 0$,then $t_{0,\beta,\gamma} = 0$ for all $\gamma \in X$. □

(3) *The case when* $\alpha \neq 0 \neq \beta$

In this case , $sg_\beta v g_\alpha ; s \in R_\beta, v \in R_\alpha$, takes the following form :

$$(*) \qquad n_a^+ g_\beta n_b^+ g_\alpha = \begin{pmatrix} a(1+b) & ab+a\alpha \\ b+\beta & b+\alpha\beta \end{pmatrix}$$

Now $(*) \in N \Leftrightarrow (b=-1 \wedge \alpha\beta = 1) \vee (\alpha = \beta = -b)$, the first case is rejected since $\alpha\beta \neq 1$, by our choice of the index set X. The second case is valid if and only if $b = \alpha$ which gives the set

$$\left\{ n_a^+ g_\beta n_{-\alpha^{-1}}^+ g_\alpha = \begin{pmatrix} a(1-\alpha) & 0 \\ 0 & \alpha+\beta \end{pmatrix} ; a \in F_q^* \right\}$$

, each member of this set contributes the coefficient $\lambda(n_a^+)\lambda(n_{-\alpha^{-1}}^+)\lambda(\begin{pmatrix} a(1-\alpha) & 0 \\ 0 & \alpha+\beta \end{pmatrix}) = 1$. Hence when we sum over all $a \in F_q^*$ we get q-1. This proves the following

PROPOSITION 4.8 Suppose that $\alpha, \beta \in X$ with $\alpha \neq 0 \neq \beta$. Then

(1) $t_{\alpha,\beta,e} \neq 0 \ \Leftrightarrow \ \alpha = \beta$,

(2) $t_{\alpha,\alpha,e} = q-1$,

(3) $t_{\alpha,\beta,\gamma} = 0$ for all $\gamma \ (\neq e) \in X$. □

Also,

$(*) \in g_0 N \Leftrightarrow [b+\alpha\beta = 0 \wedge a(b+1) = b+\beta \ ] \vee [b+\beta = 0 \wedge a(b+\alpha) = b+\alpha\beta]$

$\Leftrightarrow [b = -\alpha\beta \wedge a = {}^{\beta-\alpha\beta}\!/_{1-\alpha\beta}] \vee [\ b = -\beta \wedge a = {}^{\alpha\beta-\beta}\!/_{\alpha-\beta}]$

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & \beta \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a(b+1) & ab \\ b+\beta & b \end{pmatrix} \quad \text{or}$$

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & \beta \end{pmatrix}\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (ab+1) & a \\ b+\beta & \beta \end{pmatrix}$$

Since $b, ab, a, \beta \neq 0$, it follows that $sg_\beta v g_0 \notin N$, for all $s \in R_\beta, v \in R_0$. Therefore we have $t_{0,\beta,e} = 0$. On the other hand $\begin{pmatrix} a(b+1) & ab \\ b+\beta & b \end{pmatrix} \in g_0 N = \left\{ \begin{pmatrix} x & y \\ x & 0 \end{pmatrix}, \begin{pmatrix} y & x \\ 0 & x \end{pmatrix}; x, y \in F_q^* \right\}$ if and only if $b+\beta = 0 \wedge ab = b$, that is if and only if $b = -\beta \wedge a = 1$, in which case $\begin{pmatrix} a(b+1) & ab \\ b+\beta & b \end{pmatrix} = \begin{pmatrix} b+1 & b \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ b+1 & 0 \end{pmatrix}$, hence contributes the coefficient

4.5 $$\lambda(n_1^+)\lambda(n_{-\beta}^+)\lambda(\begin{pmatrix} 0 & -\beta \\ 1-\beta & 0 \end{pmatrix}) = -1$$

While $\begin{pmatrix} a(b+1) & a \\ b+\beta & \beta \end{pmatrix} \in g_0 N = \left\{ \begin{pmatrix} x & y \\ x & 0 \end{pmatrix}, \begin{pmatrix} y & x \\ 0 & x \end{pmatrix}; x, y \in F_q^* \right\}$ if and only if $b+\beta = 0 \wedge a = \beta$, that is if and only if $b = -\beta \wedge a = \beta$ in which case $\begin{pmatrix} a(b+1) & a \\ b+\beta & \beta \end{pmatrix} = \begin{pmatrix} -\beta^2+\beta & \beta \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \beta \\ -\beta^2+\beta & 0 \end{pmatrix}$, hence gives the coefficient

4.6 $$\lambda(n_a^+)\lambda(n_b^-)\lambda(\begin{pmatrix} 0 & \beta \\ -\beta^2+\beta & 0 \end{pmatrix}) = 1$$

By summing 4.5 and 4.6 we get $t_{0,\beta,0} = 0$. Now if $\gamma \in X \setminus \{0,1\}$, then

$$\begin{pmatrix} a(b+1) & ab \\ b+\beta & b \end{pmatrix} \in g_\gamma N = \left\{ \begin{pmatrix} x & y \\ x & \gamma y \end{pmatrix}, \begin{pmatrix} y & x \\ \gamma y & x \end{pmatrix}; x, y \in F_q^* \right\} \Leftrightarrow$$

$$\Leftrightarrow (a(b+1) = b+\beta \wedge \gamma ab = b) \vee (ab = b \wedge b+\beta = \gamma a(b+1))$$

$$\Leftrightarrow \quad (b = {}^{1-\gamma^{-1}}\!/_{\gamma^{-1}-\beta} \wedge a = \gamma^{-1}) \vee (a = 1 \wedge b = {}^{\gamma-\beta}\!/_{1-\gamma}) \quad (\text{ note that } \gamma \neq 1 \text{ })$$

$$\Leftrightarrow n_a^+ g_\beta n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix}\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \vee \quad n_a^+ g_\beta n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix}\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \text{ which}$$

contributes the coefficients +1 , -1 , respectively cancelling each other when we sum over a,b. similarly

$$\begin{pmatrix} a(b+1) & a \\ b+\beta & \beta \end{pmatrix} \in g_\gamma N = \left\{ \begin{pmatrix} x & y \\ x & \gamma y \end{pmatrix}, \begin{pmatrix} y & x \\ \gamma y & x \end{pmatrix}; x, y \in F_q^* \right\} \Leftrightarrow$$

Also $n_a^- g_0 n_b^+ g_0 \in g_0 N \Leftrightarrow b+1=0, ab=b \Leftrightarrow b=-1, a=1$, in which case $n_1^- g_0 n_{-1}^+ g_0 = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ which contributes the coefficient $\lambda(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \lambda(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}) \lambda(\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}) = 1$. Summing up the coefficients from both cases we get

PROPOSITION 4.3 $\quad t_{0,0,0} = -(q-3)$. □

Now for an arbitrary $\gamma \in F_q^*$ we have

$g_\gamma N = \left\{ \begin{pmatrix} x & y \\ x & \gamma y \end{pmatrix}, \begin{pmatrix} y & x \\ \gamma y & x \end{pmatrix}; x, y \in F_q^* \right\}$. Therefore $n_a^+ g_0 n_b^- g_0$, $n_a^- g_0 n_b^- g_0 \notin g_\gamma N$ for all $a, b \in F_q^*$, while $n_a^+ g_0 n_b^+ g_0 \in g_\gamma N$ if and only if $[ab+a=b \wedge b=\gamma ab] \vee [ab=b \wedge b=\gamma(ab+a)]$

$$\Leftrightarrow [a=\frac{b}{b+1} \wedge \gamma = \frac{b+1}{b}] \vee [a=1 \wedge \gamma = \frac{b}{b+1}] \ ; \ b \neq 0,-1$$

in which case $n_{b/b+1}^+ g_0 n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & {}^{b+1}/_{b} \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & {}^{b^2}/_{b+1} \end{pmatrix}$

and $\qquad n_1^+ g_0 n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & {}^{b}/_{b+1} \end{pmatrix} \begin{pmatrix} 0 & {}^{(b+1)^2}/_{b} \\ b+1 & 0 \end{pmatrix}$

contributing coefficients 1 and -1 , respectively, hence canceling each other when summing over all b. Similarly

$n_a^- g_0 n_b^+ g_0 \in g_\gamma N \Leftrightarrow [ab=b+1 \wedge b=\gamma(b+1)] \vee [ab=b \wedge b+1=\gamma b]$

$\Leftrightarrow [a={}^{b+1}/_{b} \wedge \gamma = {}^{b}/_{b+1}] \vee [a=1 \wedge \gamma = {}^{b+1}/_{b}]$, in which case

$n_{b+1/b}^- g_0 n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix} \begin{pmatrix} b+1 & 0 \\ 0 & b+1 \end{pmatrix} \vee n_b^- g_0 n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix} \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$,

contributing coefficients -1 , 1 ; respectively , hence canceling each other when summing over all b. This proves the following

PROPOSITION 4.4 $\quad t_{0,0,\gamma} = 0$ for all $\gamma (\neq e, 0) \in X$. □

(2) *The case when* $\alpha = 0$ *and* $\beta \neq 0$:

Consider the elements $sg_\beta v g_0$, where $s \in R_\beta$ and $v \in R_0$. By our choice of those transversal, $sg_\beta v g_0$ takes one of the following two forms

$$n_a^+ g_0 n_b^+ g_0 = \begin{pmatrix} ab+a & ab \\ b & b \end{pmatrix}$$

$$n_a^+ g_0 n_b^- g_0 = \begin{pmatrix} ab+a & a \\ b & 0 \end{pmatrix}$$

$$n_a^- g_0 n_b^- g_0 = \begin{pmatrix} ab & 0 \\ b+1 & 1 \end{pmatrix}$$

$$n_a^- g_0 n_b^+ g_0 = \begin{pmatrix} ab & ab \\ b+1 & b \end{pmatrix}$$

Now for all $a,b \in F_q^*$, we have $n_a^+ g_0 n_b^+ g_0$, $n_a^- g_0 n_b^+ g_0 \notin N$. On the other hand $n_a^+ g_0 n_b^- g_0 \in N$ if and only if $b=-1$ in which case $n_a^+ g_0 n_b^- g_0 = \begin{pmatrix} 0 & a \\ -1 & 0 \end{pmatrix}$. This contributes the coefficient $\lambda(n_a^+)\lambda(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})\lambda(\begin{pmatrix} 0 & a \\ -1 & 0 \end{pmatrix}) = 1$ for all $a \in F_q^*$ giving a total coefficient equals (q-1). Similarly $n_a^- g_0 n_b^- g_0 \in N$ if and only if $b=-1$ in which case $n_a^- g_0 n_b^- g_0 = \begin{pmatrix} -1 & 0 \\ 0 & a \end{pmatrix}$. This contributes the coefficient $\lambda(n_a^-)\lambda(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})\lambda(\begin{pmatrix} -1 & 0 \\ 0 & a \end{pmatrix}) = 1$, giving (q-1) as a total coefficient. This proves the following

PROPOSITION 4.2 $\quad t_{0,0,e} = 2(q-1)$. □

To evaluate the constant $t_{0,0,0}$, we replace N by $g_0 N$ in the above discussion and note that $g_0 N = \left\{ \begin{pmatrix} x & y \\ x & 0 \end{pmatrix}, \begin{pmatrix} y & x \\ 0 & x \end{pmatrix}; x, y \in F_q^* \right\}$. Therefore $n_a^+ g_0 n_b^+ g_0, n_a^- g_0 n_b^- g_0 \notin g_0 N$. On the other hand $n_a^+ g_0 n_b^- g_0 \in g_0 N$ if and only if $ab+a=b$ in which case $n_a^+ g_0 n_b^- g_0 = \begin{pmatrix} b & b/_{b+1} \\ b & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & b/_{b+1} \end{pmatrix}$ which contributes the following coefficient

$$\sum_{-1 \neq b \in F_q^*} [\lambda(n_{b/_{b+1}}^+)\lambda(n_b^-)\lambda(\begin{pmatrix} b & 0 \\ 0 & b/_{b+1} \end{pmatrix}) = -1\,] = -(q-2).$$

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}(N\cap {}^{g_\alpha}N) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix}(N\cap {}^{g_\alpha}N) = \begin{pmatrix} xy^{-1} & 0 \\ 0 & 1 \end{pmatrix}(N\cap {}^{g_\alpha}N),$$

and

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}(N\cap {}^{g_\alpha}N) = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} 0 & y^{-1} \\ \alpha^{-1}y^{-1} & 0 \end{pmatrix}(N\cap {}^{g_\alpha}N) = \begin{pmatrix} \alpha^{-1}xy^{-1} & 0 \\ 0 & 1 \end{pmatrix}(N\cap {}^{g_\alpha}N)$$

, we may take $R_\alpha = \left\{ n_a^+ = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}; a \in F_q^* \right\}$.

§4 THE STRUCTURE CONSTANTS OF $E(N,\lambda)$

For every $\alpha \in F_q^* \setminus \{1,-1\}$, let $a_\alpha \in E(N,\lambda)$ be the operator given by $a_\alpha([N_\lambda]) = \sum_{r\in R_\alpha} \lambda(r) r g_\alpha [N]_\lambda$. Then from the results of §2 we saw that $E(N,\lambda) = k-\mathrm{algebra}\langle a_e, a_\alpha ; \alpha \in F_q^* \setminus \{1,-1\}\rangle$, where a_e is the identity operator of $E(N,\lambda)$;that is the operator which corresponds to N. Also for all $\alpha,\beta \in F_q \setminus \{1,-1\}$ we have $a_\alpha a_\beta = \sum_{\gamma\in X} t_{\alpha,\beta,\gamma}\, a_\gamma$, where X is a subset of F_q which indexes the basis elements of $E(N,\lambda)$. Since $D_\alpha = D_{\alpha^{-1}}$ for all $\alpha \in F_q^*$, it follows that $a_\alpha = a_{\alpha^{-1}}$ and so we may choose the index set $X \subseteq F_q^*$ so that $\alpha\beta \neq 1$ for all $\alpha,\beta \in X$. We also take $0, e \in X$. From proposition 1.2 we have

4.1 $$t_{\alpha,\beta,\gamma} = \sum_{\substack{r\in R_\beta, v\in R_\alpha \\ r g_\beta v g_\alpha = g_\gamma n \in g_\gamma N}} \lambda(r)\lambda(v)\lambda(n) .$$

To determine $t_{\alpha,\beta,\gamma}$ and since R_α depends on weather $\alpha = 0$ *or* $\alpha \neq 0$ and because $E(N,\lambda)$ is commutative we only need to consider the following three cases :

(1) $\alpha = \beta = 0$, (2) $\alpha = 0$ *and* $\beta \neq 0$, (3) $\alpha \neq 0 \neq \beta$

(1)***The case when*** $\alpha = \beta = 0$: For all $a, b \in F_q^*$ we have

Hence we have the following formulae which determines the dimension of the Hecke algebra $E(N,\lambda)$.

THEOREM 2.6 $\dim_k E(N,\lambda)=\begin{cases}\frac{1}{2}(q-3)+2 & \text{if } p\neq 2\\ \frac{1}{2}(q-2)+2 & \text{if } p=2\end{cases}$ □

§3. THE TRANSVERSAL R_α

In order to find the structure constants for the algebra $E(N,\lambda)$ using the method described in §1, we need to choose a suitable transversal R_α for $\left\{x(N\cap^{g_\alpha}N);\, x\in N\right\}$, $\alpha\in F_q^*\setminus\{1,-1\}$. From 2.4 and 2.5 we need to distinguish two cases :

(1)***The case when*** $\alpha=0$. We have $N\cap^{g_0}N=\left\{\begin{pmatrix}x&0\\0&x\end{pmatrix};x\in F_q^*\right\}$. Hence

$\left|N\cap^{g_0}N\right|=q-1$ and so $|R_0|=\dfrac{2(q-1)^2}{q-1}=2(q-1)$. Since

$$\begin{pmatrix}x&0\\0&y\end{pmatrix}(N\cap^{g_0}N)=\begin{pmatrix}x&0\\0&y\end{pmatrix}\begin{pmatrix}y^{-1}&0\\0&y^{-1}\end{pmatrix}(N\cap^{g_0}N)=\begin{pmatrix}xy^{-1}&0\\0&1\end{pmatrix}(N\cap^{g_0}N),$$

and

$$\begin{pmatrix}0&x\\y&0\end{pmatrix}(N\cap^{g_0}N)=\begin{pmatrix}0&x\\y&0\end{pmatrix}\begin{pmatrix}y^{-1}&0\\0&y^{-1}\end{pmatrix}(N\cap^{g_0}N)=\begin{pmatrix}0&xy^{-1}\\1&0\end{pmatrix}(N\cap^{g_0}N),$$ we

may take $R_0=\left\{n_a^+=\begin{pmatrix}a&0\\0&1\end{pmatrix}, n_a^-=\begin{pmatrix}0&a\\1&0\end{pmatrix}; a\in F_q^*\right\}$.

(2)***The case when*** $\alpha\in F_q^*\setminus\{1,-1\}$. By 2.4(2) we have

$N\cap^{g_\alpha}N=\left\{\begin{pmatrix}x&0\\0&x\end{pmatrix},\begin{pmatrix}0&\alpha x\\x&0\end{pmatrix};x\in F_q^*\right\}$ and so $\left|N\cap^{g_\alpha}N\right|=2(q-1)$, hence

$|R_\alpha|=\dfrac{2(q-1)^2}{2(q-1)}=q-1$. Since

$$g_\alpha^{-1}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} g_\alpha = \begin{pmatrix} \dfrac{\alpha x - y}{\alpha - 1} & \dfrac{\alpha(x-y)}{\alpha - 1} \\ \dfrac{y - x}{\alpha - 1} & \dfrac{\alpha y - x}{\alpha - 1} \end{pmatrix}$$

and , $g_\alpha^{-1}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} g_\alpha = \begin{pmatrix} \dfrac{\alpha x - y}{\alpha - 1} & \dfrac{\alpha^2 x - y}{\alpha - 1} \\ \dfrac{y - x}{\alpha - 1} & \dfrac{y - \alpha x}{\alpha - 1} \end{pmatrix}$. Therefore

$g_\alpha^{-1}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} g_\alpha \in N \Leftrightarrow either\ (\alpha = -1, x = -y)\ or\ (x = y)$. Similarly

$g_\alpha^{-1}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} g_\alpha \in N \Leftrightarrow either\ (\alpha = -1, x = y)\ or\ (\alpha x = y)$.

The following determines the λ-compatible (N,N)-cosets of GL(2,q)

PROPOSITION 2.5 The coset $D_\alpha = Ng_\alpha N$ is λ-compatible if and only if $\alpha \neq -1$.

PROOF If $\alpha \neq -1$, then from 2.4 each element of $N^{g_\alpha} \cap N$ is either of the form $g_\alpha^{-1}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} g_\alpha \in N_1$ or $g_\alpha^{-1}\begin{pmatrix} 0 & x \\ \alpha x & 0 \end{pmatrix} g_\alpha \in N_2$. Therefore $\lambda^{g_\alpha}\Big|_{N^{g_\alpha} \cap N} = \lambda$ and so D_α is λ-compatible. Conversely suppose that $\alpha = -1$ then ,again by 2.4 , each element of $N^{g_{-1}} \cap N$ is of the form $g_{-1}^{-1}\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} g_{-1} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. But $\lambda^{g_{-1}}(g_{-1}^{-1}\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} g_{-1}) = \lambda(\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}) = -1$, while $\lambda(g_{-1}^{-1}\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} g_{-1}) = \lambda(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}) = 1$. Hence D_{-1} is not λ-compatible. □

If we denote by $\langle N \backslash G / N \rangle_\lambda$ the set of λ-compatible (N,N)-cosets in *GL(2,q)* then from the previous results we conclude that
$\langle N \backslash G / N \rangle_\lambda = \{N, D_0 = Ng_0N, D_\alpha = Ng_\alpha N\ (= D_{\alpha^{-1}});\ \alpha \in F_q^* \setminus \{1,-1\}\}$

The following table gives the size of each (N,N)-double cosets in GL(2,q).

D	N	D_0	D_{-1}	$D_\alpha (= D_{\alpha^{-1}}); \alpha \neq 0,1,-1$
$\lvert D \rvert$	$2(q-1)^2$	$4(q-1)^3$	$(q-1)^3$	$2(q-1)^3$

Note that
$2(q-1)^2 + 4(q-1)^3 + (q-1)^3 + (q-3)(q-1)^3 = (q-1)^2(q^2+q) = \lvert GL(2,q) \rvert$

In order to determine the dimension of the Hecke algebra $E(N,\lambda)$ we need to determine the λ-compatible (N,N)-cosets. For that purpose we first describe the subgroups $N \cap {}^{g_\alpha}N;\ \alpha \in F_q - \{1\}$.

PROPOSITION 2.4 For each $\alpha \in F_q - \{1\}$, we have

(1) If $\alpha = 0$, then $N \cap {}^{g_\alpha}N = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}; x \in F_q^* \right\}$

(2) *If* $\alpha = -1$, *then* $N \cap {}^{g_\alpha}N =$

$$\left\{ g_{-1}^{-1}\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}g_{-1},\ g_{-1}^{-1}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}g_{-1},\ g_{-1}^{-1}\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}g_{-1},\ g_{-1}^{-1}\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}g_{-1}; x \in F_q^* \right\}$$

$N \cap {}^{g_\alpha}N = \left\{ g_\alpha^{-1}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}g_\alpha,\ g_\alpha^{-1}\begin{pmatrix} 0 & x \\ \alpha x & 0 \end{pmatrix}g_\alpha;\ x \in F_q^* \right\}$ when $\alpha \neq 0,-1$.

PROOF

(1) If $x, y \in F_q^*$, then $g_0^{-1}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}g_0 = \begin{pmatrix} y & 0 \\ x-y & x \end{pmatrix} \in N \Leftrightarrow x = y$. On the other hand $g_0^{-1}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}g_0 = \begin{pmatrix} y & y \\ x-y & -y \end{pmatrix} \notin N$. Therefore we have

$N \cap {}^{g_0}N = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}; x \in F_q^* \right\}$.

(2) Now suppose that $\alpha \in F_q^* - \{1\}$. Then

PROOF Suppose that $g \in GL(2,q)$. Then the number of entries of g which equal 0 is either (1) two, (2) one or (3) none. In the first case $g \in N$ and so $NgN = N$.

(2) If $g = \begin{pmatrix} 0 & x \\ y & z \end{pmatrix}; x, y, z \in \mathbf{F}_q^*$, then $\begin{pmatrix} z & y \\ x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in NgN$

and so $NgN = D_0$. The same is true if $g = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$ or $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$.

(3) If $g = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$; $x, y, z, \in \mathbf{F}_q^* - \{1\}$ and $t \in \mathbf{F}_q - \{1\}$, then

$$\begin{pmatrix} 1 & 1 \\ 1 & y^{-1}z^{-1}xt \end{pmatrix} = \begin{pmatrix} x^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & xy^{-1} \end{pmatrix} \in NgN$$

Therefore $NgN = D_\alpha$; where $\alpha = y^{-1}z^{-1}xt$. □

DEFINITION If $g = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in GL(2,q)$, define $\pi(g) = y^{-1}z^{-1}xt$.

LEMMA 2.2 If $g' = n_1 g n_2 \in NgN$ then $\pi(g) = \pi(g')$ or $\pi(g) = \pi(g')^{-1}$. Hence $D_\alpha = D_{\alpha^{-1}}$ for all $\alpha \in \mathbf{F}_q^* - \{1\}$.

PROOF : If $n_1, n_2 \in N_1$ then $\pi(g) = \pi(g')$. If $n_1 \in N_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N_1$ and $n_2 \in N_1$ then $\pi(g') = \pi(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x) = \pi(x)^{-1} = \pi(g)^{-1}$, where $x \in NgN$.

Similarly if $n_1 \in N_1, n_2 \in N_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N_1$, then $\pi(g') = \pi(g)^{-1}$ and if n_1, n_2 are either in $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N_1$ or $N_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\pi(g) = \pi(g')$. □

The above lemma implies at once the following

PROPOSITION 2.3 (1) The Hecke algebra $E(N, \lambda)$ is commutative.
(2) If $p = 0$, then the kG-module $Y(N, \lambda)$ is multiplicity free.

PROOF (1) This follows from ([7], p.28), since $D_\alpha = D_{\alpha^{-1}}$ from lemma 2.2.

(2) See ([4], p.306 Exercise 18). □

REMARK When $\lambda = 1_H$; the trivial character of H, then the formula in 1.2 coincides with the one proved in ([7], p.15).

Now we take G=GL(n,q) ; the general linear group defined over the finite field $\mathbf{F_q}$, where **q** is a power of prime number p and let N be the set of all monomial (permutation) matrices in G. Then N is a subgroup of G in which every matrix has a unique non-zero coefficient in each row and in each column . Hence there is a group epimorphism $\mu : N \to S_n$ with $\ker \mu = T$; the set of diagonal matrices in G. Therefore we may lift the sign representation ε of S_n to a representation λ of N via μ; that is we let $\lambda(n) := \varepsilon(\mu(n))$ for all $n \in N$. We are interested in the Hecke algebra $E(N,\lambda)$ and we shall concentrate on the case when n=2.

§2. THE CASE WHEN $G=GL(2,q)$

We now take $G=GL(2,q)$, where q is a power of a prime number p. Then $N = N_1 \cup N_2$; where $N_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}; \ x, y \in F_q, xy \neq 0 \right\}$ and $N_2 = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}; \ x, y \in F_q, xy \neq 0 \right\}$. The multiplicative character $\lambda : N \to k^\times$ defined above is then given by

$$\lambda(n) = \begin{cases} 1 & \text{if } n \in N_1 \\ -1 & \text{if } n \in N_2 \end{cases}$$

Write p for the characteristic of k and note that $\lambda = 1_N$ if $p = 2$.

In order to study the structure of the Hecke algebra $E(N,\lambda)$, the first step is to find a k-basis for this algebra . As we have seen in the previous section this is equivalent to determining the set of λ-compatible (N,N)-cosets in G. First we shall determine a transversal for the set of double cosets of the subgroup N in GL(2,q) . For each $\alpha \in \mathbf{F_q} - \{1\}$, let $g_\alpha = \begin{pmatrix} 1 & 1 \\ 1 & \alpha \end{pmatrix} \in GL(2,q)$.

PROPOSITION 2.1 $\{ N, \ D_\alpha = N g_\alpha N; \alpha \in \mathbf{F_q} - \{1\} \}$ is the set of (N,N)-cosets in GL(2,q).

LEMMA 1.1 ([2], 2.2)

(1) $\dim_k(E(H,\lambda)) = |J_\lambda|$.

(2) $\{a_x, x \in J_\lambda\}$ is a k-basis of $E(H,\lambda)$. If $a_x a_y = \sum t_{x,y,z} a_z$, $(x,y,z \in J_\lambda)$, then $t_{x,y,z}$ all belong to the subring of k generated by $\lambda(H)$. □

DEFINITION If $x \in J_\lambda$, then the coset NxN is called λ*-compatible.*

The k-algebra $E(H,\lambda)$ is called the ***Hecke algebra*** associated with the triple (G,H,λ). The scalars $t_{x,y,z}$ are called the ***structure constants*** of the algebra $E(H,\lambda)$. It is known (see [9],§1.5) that any finite dimensional algebra is determined (up to isomorphism) by its structure constants. To determine the constants $t_{x,y,z}$, we note that

$$a_x a_y([H]_\lambda) = \sum_{z \in J_\lambda} t_{x,y,z} a_z([H]_\lambda)$$

$$= \sum_{z \in J_\lambda} t_{x,y,z} \sum_{s \in R_z} \lambda(s) sz[H]_\lambda$$

$$= \sum_{z \in J_\lambda} \sum_{s \in R_z} t_{x,y,z} \lambda(s) sz[H]_\lambda$$

$$= \sum_{z \in J_\lambda} (t_{x,y,z} z[H]_\lambda + \sum_{1 \neq s \in R_z} t_{x,y,z} \lambda(s) sz[H]_\lambda)$$

Therefore $t_{x,y,z} = coeffecient\ of\ z[H]_\lambda\ in\ a_x a_y([H]_\lambda)$. On the other hand

$$a_x a_y([H]_\lambda) = a_x(\sum_{z \in R_y} \lambda(r) ry[H]_\lambda)$$

$$= \sum_{r \in R_y} \lambda(r) ry a_x([H]_\lambda)$$

$$= \sum_{r \in R_y} \lambda(r) ry \sum_{v \in R_x} \lambda(v) vx[H]_\lambda$$

$$= \sum_{r \in R_y, v \in R_x} \lambda(r)\lambda(v) ryvx[H]_\lambda$$

Now the set $\{z[H]_\lambda, z \in G \setminus H\}$ is linearly independent in the group algebra kG. Therefore by comparing the coefficients we get the following .

PROPOSITION 1.2 $t_{x,y,z} = \sum_{\substack{r \in R_y, v \in R_x \\ ryvx = zh \in zH}} \lambda(r)\lambda(v)\lambda(h)$. □

[4],§67); no trace in the literature concerning the Hecke algebra $E(G,N,\lambda)$. The aim of this paper is to investigate the structure of this algebra in the case when $G=GL(2,q)$. It turns out that ,although the double cosets of N in G are not as manageable as those of B (see [4], theorem 65.4) , the generating basis for the Hecke algebra $E(G,N,\lambda)$ satisfy certain natural identities (Proposition 4.9) . We shall prove those identities in §4 after determining a standard basis (Proposition 3.5 & Theorem 3.6) and the structure constants of $E(G,N,\lambda)$ (Theorem 5.6) and use them to characterize the set of idempotents in the Hecke algebra $E(G,N,\lambda)$ (Proposition 6.3). We apply a polynomial algebra technique (§5) to those identities to construct a set of orthogonal idempotents whose sum is the identity of $E(G,N,\lambda)$. Kreig ([7], Theorem 3.4), proved that any Hecke algebra of dimension ≤ 5 is commutative. The case we consider here turns out to be commutative and hence provides an example of a commutative Hecke algebra of large dimension. Towards the end of the paper we give a partial characterization of the units in this Hecke algebra.

§1. HECKE ALGEBRAS AND THEIR STRUCTURE CONSTANTS

Let G be a finite group , H a subgroup of G, k is a field and $\lambda : H \to k^{\times}$ be a multiplicative character of H. We write $[H]_{\lambda} = \sum_{h \in H} \lambda(h^{-1})h \in kH$. It is clear that $[H]_{\lambda} h = h[H]_{\lambda} = \lambda(h)[H]_{\lambda}$, for all $h \in H$. The left ideal $kG[H]_{\lambda}$ of kG generated by $[H]_{\lambda}$, when regarded as a left kG-module, is isomorphic to the induced kG-module $Ind_H^G(L_{\lambda})$, where L_{λ} is a one-dimensional kH-module which affords λ. If $x \in G$, write $H^x = x^{-1}Hx$ and let λ^x be the multiplicative character of H^x given by $\lambda^x(x^{-1}hx) = \lambda(h)$ for all $h \in H$. Let $H \backslash G / H = \{ D_x := HxH \ ; x \in I \}$ be the set of distinct (H,H)-double cosets of H in G and let $J_{\lambda} = \{ x \in I \ ; \lambda^x = \lambda \ \ on \ H^x \cap H \}$. Write $Y(H,\lambda) = kG[H]_{\lambda}$ and let $E(H,\lambda) = End_{kG}(Y(H,\lambda))$; the endomorphism algebra of the kG-module $Y(H,\lambda)$. If $x \in J_{\lambda}$, write $H = \bigcup_{r \in R_x} r(H \cap {}^xH)$ and assume that $1 \in R_x$ where 1 is the identity of G. Define $a_x \in E(H,\lambda)$ as follows :

$$a_x([H]_{\lambda}) = \sum_{r \in R_x} \lambda(r) rx[H]_{\lambda}$$

THE STRUCTURE OF A HECKE ALGEBRA FOR THE GENERAL LINEAR GROUPS

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Abstract: *We study the structure of the Hecke algebra $E(N,\lambda)$ where N is the monomial matrices of degree two over a finite field and λ is the multiplicative character of N lifted from the sign character of the symmetric group. We give a standard basis and determine the structure constants for this Hecke algebra. We also use this presentation together with a polynomial algebra technique to construct an orthogonal idempotent decomposition in $E(N,\lambda)$ and partially characterize its units. The set of idempotents in this Hecke algebra is also characterized.*

Keywords: *Hecke algebra, Structure constants*

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§0. INTRODUCTION

Hecke algebras play a very important role in the representation of finite groups. One of the most striking examples that show the significance of Hecke algebras in this manner is the Hecke algebra $E(G,B,\mathbf{1_B})$, associated with the triple $(G,B,\mathbf{1_B})$ where G is a finite group of Lie type ,B is a Borel subgroup of G and $\mathbf{1_B}$ is the trivial character of B . Every such group has a structure of split BN-pair (G,B,N,R,U) (see for example [1] ,[3] ,[5],[6],[10]). Let $G=GL(n,q)$; the general linear group with coefficient taken from a finite field $\mathbf{F_q}$,where q is a power of some prime , with its standard split BN-pair (see [4] , §65B) , where B is taken to be the set of upper triangular matrices and N the set of monomial matrices. Then N is an extension of the symmetric group S_n and as such it has a multiplicative character λ given by lifting the sign character of S_n. Although the Hecke algebra $E(G,B,\mathbf{1_B})$ is a well-studied subject (see for instance [5] and

بسم الله الرحمن الرحيم

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